

BOUND STATES FOR RAPIDLY OSCILLATORY SCHRÖDINGER OPERATORS IN DIMENSION 2

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ABSTRACT. We study the eigenvalues of Schrödinger operators $-\Delta_{\mathbb{R}^2} + V_\varepsilon$ on \mathbb{R}^2 with rapidly oscillatory potential $V_\varepsilon(x) = W(x, x/\varepsilon)$, where $W(x, y) \in C_0^\infty(\mathbb{R}^2 \times \mathbb{T}^2)$ satisfies $\int_{\mathbb{T}^2} W(x, y) dy = 0$. We show that for ε small enough, such operators have a unique negative eigenvalue, that is exponentially close to 0.

1. INTRODUCTION.

We study the L^2 -eigenvalues of the Schrödinger operator $-\Delta_{\mathbb{R}^2} + V_\varepsilon$ on \mathbb{R}^2 , where ε is a small parameter and V_ε is a real-valued, compactly supported, and rapidly oscillatory potential on \mathbb{R}^2 :

$$V_\varepsilon(x) = W(x, x/\varepsilon), \quad W(x, y) = \sum_{k \in \mathbb{Z}^2 \setminus 0} W_k(x) e^{iky}.$$

The functions W_k are smooth, with support in $\mathbb{B}(0, L) = \{x \in \mathbb{R}^2, |x| < L\}$, and satisfy $\overline{W_k} = W_{-k}$. The potential V_ε is a first approximation to model disordered medias with scale of heterogeneity $\sim \varepsilon$ – oscillations play here the role of randomness. This note provides a simple proof of a conjecture of Duchêne–Vukićević–Weinstein [DVW14]:

Theorem 1. *For ε small enough, the operator $-\Delta_{\mathbb{R}^2} + V_\varepsilon$ has a unique bound state, with energy E_ε given by*

$$E_\varepsilon = -\exp\left(-\frac{4\pi}{\varepsilon^2 \int_{\mathbb{R}^2} \Lambda_0(x) dx + o(\varepsilon^2)}\right), \quad \Lambda_0(x) \stackrel{\text{def}}{=} \sum_{k \neq 0} \frac{|W_k(x)|^2}{|k|^2}. \quad (1.1)$$

In one dimension, the study of the spectral properties of $-\partial_x^2 + V_\varepsilon$ with V_ε rapidly oscillatory originated with Borisov–Gadył'shin [BoGa06], who gave a sufficient condition for the existence of a bound state, and derived the asymptotic of the corresponding energy:

$$E_\varepsilon = -\frac{\varepsilon^4}{4} \int_{\mathbb{R}} \Lambda_0(x) dx, \quad \Lambda_0(x) \stackrel{\text{def}}{=} \sum_{k \neq 0} \frac{|W_k(x)|^2}{|k|^2}. \quad (1.2)$$

This study was continued later in Borisov [Bo07] then Duchêne–Weinstein [DW11], in particular to include less regular potentials.

Duchêne–Vukićević–Weinstein [DVW14] studied the behavior of scattering quantities of Schrödinger operators with potentials which are the sum of a slowly varying term

W_0 and a rapidly oscillatory term V_ε . They showed that the transmission coefficient of the *effective potential* $W_0 - \varepsilon^2 \Lambda_0$, which is a $O(\varepsilon^2)$ -perturbation of W_0 , differs from the one of $W_0 + V_\varepsilon$ by $O(\varepsilon^3)$. When $W_0 = 0$, they recovered the result of [BoGa06]: $-\partial_x^2 + V_\varepsilon$ has a unique negative eigenvalue satisfying (1.2) for small ε . They proved a uniform weighted dispersive estimate for the propagator $e^{it(-\partial_x^2 + V_\varepsilon)}$ as $\varepsilon \rightarrow 0$, despite the presence of an eigenvalue near the edge of the continuous spectrum.

Recently, Duchêne–Raymond [DR16] obtained homogenization results for potentials of the form $\varepsilon^{-\beta} V_\varepsilon$, $\beta \in (0, 2)$, in dimension 1, using a normal form approach. Dimassi [Di16] applied ε -semiclassical calculus to show a trace formula and a Weyl law for potentials of the form $\varepsilon^{-2} V_\varepsilon$, in any dimension d .

Motivated by [DVW14] and by Christiansen [Ch06], we used a different approach to study in [Dr15] the resonances and eigenvalues of $-\Delta_{\mathbb{R}^d} + W_0 + V_\varepsilon$, in any odd dimension d . When $W_0 = 0$, we proved that the resonances and eigenvalues of V_ε escape all bounded regions as $\varepsilon \rightarrow 0$ (except the one converging to 0 when $d = 1$). When $W_0 \neq 0$, we showed that the resonances and eigenvalues of $W_0 + V_\varepsilon$ converge to the one of W_0 , with a complete expansion in powers of ε . We improved upon the homogenization result of [DVW14], refining the effective potential $W_0 - \varepsilon^2 \Lambda_0$ to $W_0 - \varepsilon^2 \Lambda_0 - \varepsilon^3 \Lambda_1$, and deriving it for any odd d . We refer to [Dr15, Figure 2] for numerical results and to [Dr15, §1] for additional references.

A famous result of Simon [Si76, Theorem 3.4] in dimension 2 predicts that under suitable conditions, a Schrödinger operator with small negative potential $-\Delta_{\mathbb{R}^2} + \epsilon \Lambda$ has a unique bound state with energy

$$E_\epsilon = -\exp\left(-\frac{4\pi}{\epsilon \int_{\mathbb{R}^2} \Lambda(x) dx + o(\epsilon)}\right). \quad (1.3)$$

This identity put together with (1.1) supports the main idea of [DVW14]: the scattering quantities of rapidly oscillatory potentials are similar to the one of suitable small potentials.

To prove Theorem 1, we first follow [Si76, §3]: we use a modified Fredholm determinant to reduce the study of eigenvalues of $-\Delta_{\mathbb{R}^2} + V_\varepsilon$ to an equation involving a certain trace. Then, we provide estimates on this trace following some ideas of [Dr15], ending the proof. We mention that the result of Theorem 1 still applies when W is not smooth but satisfies instead the weaker bound

$$\sum_{k \neq 0} |W_k|_\infty + \frac{|W_k|_{C^1}}{|k|} + \frac{|W_k|_{C^2}}{|k|^2} + \frac{|W_k|_{C^3}}{|k|^3} + \sum_{0 \neq k \neq \ell} \frac{|W_k|_{C^3} |W_\ell|_{C^3}}{|k - \ell|^{5/2}} < \infty,$$

with no change in the proof.

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Notations. We will use the following:

- For $x \in \mathbb{R}^2$, $\langle x \rangle$ denotes the Japanese bracket of x : $\langle x \rangle \stackrel{\text{def}}{=} (1 + |x|^2)^{1/2}$.
- The function $z \mapsto \ln(z)$ denotes the holomorphic logarithm on $\mathbb{C} \setminus (-\infty, 0]$.
- $C_0^\infty(\mathbb{R}^2, \mathbb{R})$ is the set of real smooth compactly supported functions on \mathbb{R}^2 .
- L^2 denotes the Hilbert space of functions that are squared integrable, with L^2 -norm denoted by $|f|_2$ and scalar product given by $\langle f, g \rangle_2 = \int_{\mathbb{R}^2} \bar{f}g$.
- C^k is the space of functions on \mathbb{R}^2 with k continuous and bounded derivatives, with norm $|f|_{C^k} = |f|_\infty + |f'|_\infty + \dots + |f^{(k)}|_\infty$.
- We write Δ for the Laplacian $\Delta_{\mathbb{R}^2}$, and \hat{f} for the Fourier transform of f : $\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\xi} f(x) dx$. H^s denotes the standard Sobolev space on \mathbb{R}^2 , with norm $|f|_{H^s} = |\langle \xi \rangle^s \hat{f}|_2 = |(\text{Id} - \Delta)^{s/2} f|_2$.
- \mathcal{B} is the Banach space of bounded linear operators from L^2 to itself.
- \mathcal{L}^2 is the Banach space of Hilbert–Schmidt operators on L^2 .

2. GENERAL PROPERTIES.

Let $R_0(\lambda, x, y)$ be the kernel of the free resolvent $(-\Delta + \lambda^2)^{-1}$. It is a Hankel function of $\lambda|x - y|$ and one can write

$$R_0(\lambda, x, y) = -\frac{1}{2\pi} \ln(\lambda) + H_0(\lambda, x, y),$$

where for every $x \neq y$ the function $\lambda \mapsto H_0(\lambda, x, y)$ is holomorphic in $\{\text{Re } \lambda > 0\}$ and admits a continuous extension to $\{\text{Re } \lambda \geq 0\}$ – see [Si76, (12)]. For λ with $\text{Re } \lambda \geq 0$, let $H_0(\lambda)$ be the operator with kernel $H_0(\lambda, x, y)$. When $\lambda \in (0, \infty)$, $R_0(\lambda)$ is selfadjoint and $\ln(\lambda)$ is real, hence $H_0(\lambda)$ is selfadjoint.

Let $\mathcal{V} \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$, with support in $\mathbb{B}(0, L) = \{x \in \mathbb{R}^2, |x| \leq L\}$, and $\rho \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$, equal to 1 on the support of \mathcal{V} and 0 outside $\mathbb{B}(0, L)$. We will be interested in the behavior of $K_{\mathcal{V}}(\lambda) = \rho R_0(\lambda) \mathcal{V}$ for λ close to 0. Write $K_{\mathcal{V}}(\lambda) = \Pi_{\mathcal{V}} \ln(\lambda) + L_{\mathcal{V}}(\lambda)$,

$$\Pi_{\mathcal{V}} \stackrel{\text{def}}{=} -\frac{1}{2\pi} \rho \otimes \mathcal{V}, \quad L_{\mathcal{V}}(\lambda) \stackrel{\text{def}}{=} \rho H_0(\lambda) \mathcal{V}, \quad \text{Re } \lambda \geq 0.$$

An operator $A : C_0^\infty(\mathbb{R}^2) \rightarrow \mathcal{D}'(\mathbb{R}^2)$ belongs to \mathcal{L}^2 (the Hilbert–Schmidt class) if and only if its kernel $A(x, y)$ is in $L^2(\mathbb{R}^4, dxdy)$. In this case the \mathcal{L}^2 -norm is

$$|A|_{\mathcal{L}^2} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} |A(x, y)|^2 dxdy \right)^{1/2}.$$

We prove the following result, similar to [Si76, Proposition 3.2]:

Lemma 2.1. *The operator $L_\rho(\lambda)$ is in \mathcal{L}^2 . Moreover, uniformly locally in λ, μ with nonnegative real parts,*

$$|L_\rho(\lambda) - L_\rho(\mu)|_{\mathcal{B}} = O(|\lambda \ln(\lambda) - \mu \ln(\mu)|), \quad |L_\rho(\lambda)|_{L^2 \rightarrow H^2} = O(1). \quad (2.1)$$

Proof. To prove the first part of (2.1), we write the kernel of $H_0(\lambda)$ as

$$H_0(\lambda, x, y) = -\frac{1}{2\pi} \ln|x - y| + F(\lambda|x - y|).$$

The function $F : \mathbb{C} \rightarrow \mathbb{C}$ takes the form $F(\zeta) = \zeta h(\zeta) \ln(\zeta) + g(\zeta)$ for some entire functions g, h – see [Si76, (13)]. For $x \in \text{supp}(\rho), y \in \text{supp}(\rho)$ and λ in a compact set, $\lambda|x - y|$ is uniformly bounded. Hence, there exists a constant C such that for such values of x, y, μ, λ ,

$$|F(\lambda|x - y|) - F(\mu|x - y|)| \leq C|\lambda \ln(\lambda) - \mu \ln(\mu)| \cdot |\ln|x - y||.$$

The bound $|L_\rho(\lambda) - L_\rho(\mu)|_{\mathcal{B}} = O(|\lambda \ln(\lambda) - \mu \ln(\mu)|)$ follows now from Schur's test and from the local integrability of $\ln|x - y|$ on \mathbb{R}^2 .

The operator $L_\rho(\lambda)$ belongs to \mathcal{L}^2 by [Si76, Proposition 3.2]. The second estimate of (2.1) amounts to prove that $\Delta L_\rho(\lambda)$ belongs to \mathcal{B} , uniformly locally for $\lambda \in K$. By the same argument as in [DyZw16, Theorem 2.1], it suffices to show that $\rho \Delta H_0(\lambda) \rho$ belongs to \mathcal{B} . Recall that $\Delta \ln|x| = -2\pi\delta$, so that

$$\Delta_x H_0(\lambda, x, y) = \delta(x - y) + \Delta_x(F(\lambda|x - y|)) \quad (2.2)$$

We now compute the Laplacian of $F(\lambda|x - y|)$ with respect to x . Note that

$$\Delta(g(\lambda|z|)) = 4 \frac{\partial^2 g(\lambda|z|)}{\partial z \partial \bar{z}} = \frac{\lambda}{|z|} g'(\lambda|z|) + \lambda^2 g''(\lambda|z|). \quad (2.3)$$

Define $f(\lambda|z|) \stackrel{\text{def}}{=} \lambda|z|h(\lambda|z|)$. By applying (2.3) to $f \cdot \ln$ instead of g , we get

$$\begin{aligned} \Delta(f(\lambda|z|) \ln(\lambda|z|)) &= -2\pi f(\lambda|z|)\delta + \frac{2f'(\lambda|z|)}{\lambda|z|} + \left(\frac{\lambda f'(\lambda|z|)}{|z|} + \lambda^2 f''(\lambda|z|) \right) \ln(\lambda|z|) \\ &= \frac{2f'(\lambda|z|)}{|z|} + \left(\frac{\lambda f'(\lambda|z|)}{|z|} + \lambda^2 f''(\lambda|z|) \right) \ln(\lambda|z|). \end{aligned} \quad (2.4)$$

In the second line we used that the product of a smooth function vanishing at 0 with δ vanishes. The right hand sides of (2.3) and (2.4) both define locally integrable functions of $z \in \mathbb{C}$, uniformly locally in λ . Since ρ is compactly supported, Schur's test combined with (2.2) shows that $\rho \Delta H_0(\lambda) \rho$ belongs to \mathcal{B} uniformly in λ . This concludes the proof. \square

Since $L_{\mathcal{V}}(\lambda)$ is in \mathcal{L}^2 , we can define the modified Fredholm determinant $d_{\mathcal{V}}(\lambda)$ by

$$d_{\mathcal{V}}(\lambda) \stackrel{\text{def}}{=} \text{Det}(\text{Id} + \Psi(L_{\mathcal{V}}(\lambda))), \quad \Psi(z) \stackrel{\text{def}}{=} (1 + z)e^{-z} - 1, \quad \text{Re}(\lambda) \geq 0,$$

see [DyZw16, Appendix B]. As $K_{\mathcal{V}}(\lambda)$ is the sum of the rank one operator $\Pi_{\mathcal{V}} \ln(\lambda)$ with $L_{\mathcal{V}}(\lambda)$, $K_{\mathcal{V}}(\lambda)$ belongs to \mathcal{L}^2 . We define

$$D_{\mathcal{V}}(\lambda) \stackrel{\text{def}}{=} \text{Det}(\text{Id} + \Psi(K_{\mathcal{V}}(\lambda))), \quad \text{Re}(\lambda) > 0.$$

The negative eigenvalues of $-\Delta + \mathcal{V}$ are exactly the numbers of the form $-\lambda^2$, where $\lambda \in (0, \infty)$ is a zero of $D_{\mathcal{V}}(\lambda)$, see [GLMZ05, Theorem 5.4]. The next result is similar to [Si76, Theorem 3.3].

Lemma 2.2. *Let λ such that $\text{Re } \lambda > 0$ and $\text{Id} + L_{\mathcal{V}}(\lambda)$ is invertible on L^2 . Then,*

$$D_{\mathcal{V}}(\lambda) = 0 \Leftrightarrow 1 + \ln(\lambda)\varphi_{\mathcal{V}}(\lambda) = 0, \quad \varphi_{\mathcal{V}}(\lambda) \stackrel{\text{def}}{=} \text{Tr}((\text{Id} + L_{\mathcal{V}}(\lambda))^{-1} \Pi_{\mathcal{V}}).$$

If moreover $\lambda \in (0, \infty)$ then $\varphi_{\mathcal{V}}(\lambda) \in \mathbb{R}$.

Proof. To simplify the notations of this proof, we simply write $L_{\mathcal{V}}, K_{\mathcal{V}}$ for the operators $L_{\mathcal{V}}(\lambda), K_{\mathcal{V}}(\lambda)$. Let λ with $\operatorname{Re} \lambda > 0$ and $\operatorname{Id} + L_{\mathcal{V}}$ invertible on L^2 . Since $K_{\mathcal{V}} = \Pi_{\mathcal{V}} \ln(\lambda) + L_{\mathcal{V}}$ we have

$$\operatorname{Id} + K_{\mathcal{V}} = (\operatorname{Id} + L_{\mathcal{V}}) (\operatorname{Id} + (\operatorname{Id} + L_{\mathcal{V}})^{-1} \Pi_{\mathcal{V}} \ln(\lambda)). \quad (2.5)$$

Recall that $1 + z = (1 + \Psi(z))e^z$ so that for every bounded operator A , $\operatorname{Id} + A = (\operatorname{Id} + \Psi(A))e^A$. This identity combined with (2.5) shows that

$$e^{K_{\mathcal{V}}} (\operatorname{Id} + \Psi(K_{\mathcal{V}})) = e^{L_{\mathcal{V}}} (\operatorname{Id} + \Psi(L_{\mathcal{V}})) (\operatorname{Id} + (\operatorname{Id} + L_{\mathcal{V}})^{-1} \Pi_{\mathcal{V}} \ln(\lambda)). \quad (2.6)$$

We observe that $K_{\mathcal{V}} - L_{\mathcal{V}} = \ln(\lambda) \Pi_{\mathcal{V}}$. Since $\Pi_{\mathcal{V}}$ has kernel $-\rho \otimes \mathcal{V}/(2\pi)$, its trace is equal to $\alpha = -\int_{\mathbb{R}^2} \mathcal{V}/(2\pi)$. Taking the determinant on both sides of (2.6) we obtain

$$\lambda^{\alpha} D_{\mathcal{V}}(\lambda) = d_{\mathcal{V}}(\lambda) \cdot \operatorname{Det} (\operatorname{Id} + (\operatorname{Id} + L_{\mathcal{V}})^{-1} \Pi_{\mathcal{V}} \ln(\lambda)).$$

Since the operator $(\operatorname{Id} + L_{\mathcal{V}})^{-1} \Pi_{\mathcal{V}} \ln(\lambda)$ is of rank one, the determinant of $\operatorname{Id} + (\operatorname{Id} + L_{\mathcal{V}})^{-1} \Pi_{\mathcal{V}} \ln(\lambda)$ is equal to $1 + \operatorname{Tr} ((\operatorname{Id} + L_{\mathcal{V}})^{-1} \Pi_{\mathcal{V}} \ln(\lambda))$. Defining $\varphi_{\mathcal{V}}(\lambda) = \operatorname{Tr} ((\operatorname{Id} + L_{\mathcal{V}})^{-1} \Pi_{\mathcal{V}})$, we obtain

$$\lambda^{\alpha} D_{\mathcal{V}}(\lambda) = d_{\mathcal{V}}(\lambda) \cdot (1 + \ln(\lambda) \varphi_{\mathcal{V}}(\lambda)). \quad (2.7)$$

Since $\operatorname{Re} \lambda > 0$ we have $\lambda \neq 0$ and the first part of the lemma follows.

For the second part of the lemma, it suffices to prove that $D_{\mathcal{V}}(\lambda)$ and $d_{\mathcal{V}}(\lambda)$ are both real when $\lambda > 0$, thanks to (2.7). For such λ , $R_0(\lambda)$ is selfadjoint. Hence,

$$\overline{D_{\mathcal{V}}(\lambda)} = \operatorname{Det}(\operatorname{Id} + \Psi(K_{\mathcal{V}}^*)) = \operatorname{Det}(\operatorname{Id} + \Psi(\mathcal{V} R_0(\lambda) \rho)).$$

Since Ψ vanishes at 0, there exists ψ entire with $\Psi(z) = z\psi(z)$. Therefore,

$$\overline{D_{\mathcal{V}}(\lambda)} = \operatorname{Det}(\operatorname{Id} + \mathcal{V} R_0(\lambda) \rho \psi(\mathcal{V} R_0(\lambda) \rho)) = \operatorname{Det}(\operatorname{Id} + \rho R_0(\lambda) \rho \cdot \psi(\mathcal{V} R_0(\lambda) \rho) \mathcal{V}). \quad (2.8)$$

In the last equality, we used that if $B \in \mathcal{B}$ and A is trace-class then $\operatorname{Det}(\operatorname{Id} + BA) = \operatorname{Det}(\operatorname{Id} + AB)$, see [DyZw16, (B.5.13)]. The power series expansion of $\psi(z) = \sum_{m=0}^{\infty} a_m z^m$ implies that $\rho R_0(\lambda) \rho \cdot \psi(\mathcal{V} R_0(\lambda) \rho) \mathcal{V}$

$$= \rho R_0(\lambda) \rho \sum_{m=0}^{\infty} a_m (\mathcal{V} R_0(\lambda) \rho)^m \mathcal{V} = \sum_{m=0}^{\infty} a_m (\rho R_0(\lambda) \mathcal{V})^{m+1} = \Psi(L_{\mathcal{V}}).$$

Equation (2.8) now shows that $D_{\mathcal{V}}(\lambda) = \overline{D_{\mathcal{V}}(\lambda)}$ when $\lambda \in (0, \infty)$. The same arguments (using that $H_0(\lambda)$ is selfadjoint for $\lambda \in (0, \infty)$) shows that $d_{\mathcal{V}}(\lambda) \in \mathbb{R}$ if $\lambda \in (0, \infty)$. Hence, (2.7) shows that $\varphi_{\mathcal{V}}(\lambda) \in \mathbb{R}$ when $\lambda \in (0, \infty)$. \square

3. BOUND STATE EXPONENTIALLY CLOSE TO ZERO.

We now focus on the case of a potential $V = V_{\varepsilon}$ given by

$$V_{\varepsilon}(x) \stackrel{\text{def}}{=} W\left(x, \frac{x}{\varepsilon}\right), \quad W(x, y) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}^2 \setminus 0} W_k(x) e^{iky}.$$

Here the functions $W_k \in C^\infty(\mathbb{R}^2)$ have supports in $\mathbb{B}(0, L) = \{x \in \mathbb{R}^2, |x| < L\}$, and $\overline{W_k} = W_{-k}$. To simplify notations, we will drop the index ε in V_ε and write $V = V_\varepsilon$. We first investigate the invertibility of $\text{Id} + L_V(\lambda)$ for ε small enough. We will need the following lemma regarding the behavior as $\varepsilon \rightarrow 0$ of certain oscillatory integrals:

Lemma 3.1. *The following estimates hold:*

$$|V|_{H^{-2}} = O(\varepsilon^2), \quad \left| \langle D \rangle^{-2} V \langle D \rangle^{-2} \right|_{\mathcal{B}} = O(\varepsilon^2), \quad \langle D \rangle \stackrel{\text{def}}{=} (\text{Id} - \Delta)^{1/2}.$$

Proof. We estimate $|V|_{H^{-2}}$ in a similar way as in the proof of [Dr15, Theorem 1]:

$$\begin{aligned} |V|_{H^{-2}} &= \left| \langle \xi \rangle^{-2} \sum_{k \neq 0} \widehat{W_k}(\xi - k/\varepsilon) \right|_2 \leq \sum_{k \neq 0} \left| \langle \xi \rangle^{-2} \langle \xi - k/\varepsilon \rangle^{-2} \langle \xi - k/\varepsilon \rangle^2 \widehat{W_k}(\xi - k/\varepsilon) \right|_2 \\ &\leq \sum_{k \neq 0} \left| \langle \xi \rangle^{-2} \langle \xi - k/\varepsilon \rangle^{-2} \right|_\infty \cdot |W_k|_{H^2} \leq C \sum_{k \neq 0} \langle k/\varepsilon \rangle^{-2} |W_k|_{H^2} = O(\varepsilon^2). \end{aligned}$$

In the last line we used Peetre's inequality: for $t > 0$, $x, y \in \mathbb{R}^2$,

$$\langle x \rangle^{-t} \langle y \rangle^{-t} \leq 2^t \langle x - y \rangle^{-t}. \quad (3.1)$$

This shows the first estimate.

The boundedness of $\langle D \rangle^{-2} V \langle D \rangle^{-2}$ on L^2 is equivalent to the boundedness of the multiplication operator by V from H^2 to H^{-2} . We recall that H^2 is an algebra in dimension 2, and that there exists $C > 0$ such that for any $u, f \in H^2$, $|fu|_{H^2} \leq C|f|_{H^2}|u|_{H^2}$. The multiplication operator by $u \in H^2$ is bounded from H^2 to H^2 , hence from H^{-2} to H^{-2} with same norm and we deduce the inequality $|fu|_{H^{-2}} \leq C|f|_{H^{-2}}|u|_{H^2}$. Applying this with $V = f$ shows that $|V|_{H^2 \rightarrow H^{-2}} \leq C|V|_{H^{-2}} = O(\varepsilon^2)$. This ends the proof \square

Lemma 3.2. *For every compact subset K of $\{\lambda : \text{Re}(\lambda) \geq 0\}$, there exist $C, \varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, the inverse of $\text{Id} + L_V(\lambda)$ exist, is given by a convergent Neumann series, and satisfies*

$$|(\text{Id} + L_V(\lambda))^{-1}|_{\mathcal{B}} \leq C.$$

Proof. To prove the lemma it suffices to show that $L_V(\lambda)^2$ is bounded with $|L_V(\lambda)^2|_{\mathcal{B}} < \frac{1}{2}$, for ε small enough. Recall that $V \in L^\infty$; and (Lemma 2.1) that for $\lambda \in K$, $L_\rho(\lambda)$ is uniformly bounded from L^2 to H^2 (hence by the adjoint bound, from H^{-2} to L^2). These facts imply

$$|L_V(\lambda)^2|_{\mathcal{B}} \leq C|L_\rho(\lambda)V L_\rho(\lambda)|_{\mathcal{B}} \leq C|\langle D \rangle^{-2} V \langle D \rangle^{-2}|_{\mathcal{B}}.$$

The RHS is $O(\varepsilon^2)$ by Lemma 3.1. This ends the proof. \square

The negative eigenvalues $-\lambda^2$ of the operator $-\Delta + V$ on L^2 all belong to a fixed compact set: if $-\lambda^2$ is a negative eigenvalue, then there exists $0 \neq u \in L^2$ such that

$$-\Delta u + Vu + \lambda^2 u = 0.$$

Multiplying by \bar{u} on both sides and integrating by parts, we obtain

$$|\nabla u|_2^2 + \langle Vu, u \rangle_2 + \lambda^2 |u|_2^2 = 0.$$

Thus, $\lambda^2 |u|_2^2 \leq |V|_\infty |u|_2^2$, which shows that $-\lambda^2 \in [-|V|_\infty, 0] \subset [-M^2, 0]$ where $M = 2 + \sum_{k \neq 0} |W_k|_\infty$ is independent of ε .

This fact, together with Lemma 2.2 and 3.2 (applied with $K = K_0 \stackrel{\text{def}}{=} [0, M]$), implies that for ε small enough the set of negative eigenvalues of $-\Delta + V$ is exactly $\{-\lambda^2\}$, where λ solves the equation

$$1 + \ln(\lambda) \varphi_V(\lambda) = 0, \quad \lambda \in K_0, \quad \varphi_V(\lambda) \stackrel{\text{def}}{=} \text{Tr}((\text{Id} + L_V(\lambda))^{-1} \Pi_V).$$

Lemma 3.3. *Uniformly for λ, μ in $K_0 = [0, M]$,*

$$|\varphi_V(\lambda)| = O(\varepsilon^2), \quad |\varphi_V(\lambda) - \varphi_V(\mu)| = O(|\lambda \ln(\lambda) - \mu \ln(\mu)|). \quad (3.2)$$

Proof. We work exclusively with $\lambda, \mu \in K_0$ and ε small enough so that Lemma 3.2 applies. All the estimates below are uniform for such λ, ε . Write $(\text{Id} + L_V(\lambda))^{-1} = \text{Id} - L_V(\lambda)(\text{Id} + L_V(\lambda))^{-1}$ to get

$$\begin{aligned} \varphi_V(\lambda) &= \text{Tr}(\Pi_V) - \text{Tr}(L_V(\lambda)(\text{Id} + L_V(\lambda))^{-1} \Pi_V) = \frac{-\langle \rho, V \rangle_2 + \langle L_V(\lambda) f_\lambda, V \rangle_2}{2\pi}, \\ f_\lambda &\stackrel{\text{def}}{=} (\text{Id} + L_V(\lambda))^{-1} \rho. \end{aligned} \quad (3.3)$$

The term $\langle \rho, V \rangle_2$ is clearly independent of λ and equal to $O(\varepsilon^\infty)$. Let $P_k = (k_1 D_{x_1} + k_2 D_{x_2})/|k|^2$, so that $P_k e^{ikx/\varepsilon} = \varepsilon e^{ikx/\varepsilon}$. Since P_k is selfadjoint, $\langle L_V(\lambda) f_\lambda, V \rangle_2 =$

$$\varepsilon^2 \sum_{k \neq 0} \langle W_k \cdot L_V(\lambda) f_\lambda, \rho P_k^2 e^{-ik\bullet/\varepsilon} \rangle_2 = \varepsilon^2 \sum_{k \neq 0} \langle P_k^2 (W_k \cdot L_V(\lambda) f_\lambda), \rho e^{-ik\bullet/\varepsilon} \rangle_2.$$

The Cauchy–Schwarz inequality yields

$$|\varphi_V(\lambda)| \leq O(\varepsilon^\infty) + C\varepsilon^2 \sum_{k \neq 0} |P_k^2 (W_k \cdot L_V(\lambda) f_\lambda)|_2.$$

By Lemma 3.2, the operator $(\text{Id} + L_V(\lambda))^{-1}$ is uniformly bounded in \mathcal{B} , hence $|f_\lambda|_2 = O(1)$. Since $L_V(\lambda)$ maps L^2 to H^2 the operator $P_k^2 W_k L_V(\lambda)$ maps L^2 to L^2 with

$$|P_k^2 W_k L_V(\lambda)|_{\mathcal{B}} \leq C \frac{|W_k|_{C^2}}{|k|^2} |V|_\infty.$$

Thus $|\varphi_V(\lambda)| = O(\varepsilon^2)$ as claimed.

Regarding the second estimate in (3.2), we use (3.3) and the Cauchy–Schwarz inequality to see that

$$\begin{aligned} |\varphi_V(\lambda) - \varphi_V(\mu)| &\leq \frac{1}{2\pi} |L_V(\lambda) f_\lambda - L_V(\mu) f_\mu|_2 |V|_2 \\ &\leq C |L_\rho(\lambda) - L_\rho(\mu)|_{\mathcal{B}} + C |f_\lambda - f_\mu|_2 \leq C |L_\rho(\lambda) - L_\rho(\mu)|_{\mathcal{B}}. \end{aligned}$$

In the above we used $f_\lambda = (\text{Id} + L_V(\lambda))^{-1} \rho$, the resolvent identity, and the uniform boundedness of $(\text{Id} + L_V(\lambda))^{-1}$. The conclusion follows now from Lemma 2.1. \square

We now identify $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \varphi_V(\lambda)$:

Lemma 3.4. *Uniformly for $\lambda \in K_0$,*

$$\varphi_V(\lambda) = \frac{\varepsilon^2}{2\pi} \int_{\mathbb{R}^2} \Lambda_0(x) dx + o(\varepsilon^2), \quad \Lambda_0(x) \stackrel{\text{def}}{=} \sum_{k \neq 0} \frac{|W_k(x)|^2}{|k|^2}. \quad (3.4)$$

Proof. 1. We claim that it is enough to show (3.4) for $\lambda \in [1, 2]$. Indeed, Lemma 3.3 shows that $\varepsilon^{-2} \varphi_V$ is uniformly bounded. In addition, it is a holomorphic function of $\lambda \in K_0$. Hence, after possibly passing to a subsequence, it converges uniformly locally to a holomorphic function. If (3.4) holds for $\lambda \in [1, 2]$, $\varepsilon^{-2} \varphi_V$ has only one accumulation point on $[1, 2]$, hence – by the unique continuation principle – only one accumulation point on K_0 . This would prove that (3.4) holds uniformly on K_0 . Below we show (3.4) for $\lambda \in [1, 2]$. We will always assume that ε is small enough so that Lemma 3.2 holds.

2. By expanding $(\text{Id} + L_V(\lambda))^{-1}$ into a finite Born series,

$$\begin{aligned} \varphi_V(\lambda) &= \text{Tr}(\Pi_V) - \text{Tr}(L_V(\lambda)\Pi_V) + \text{Tr}(L_V(\lambda)(\text{Id} + L_V(\lambda))^{-1}L_V(\lambda)\Pi_V) \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} V(x) dx + \frac{1}{2\pi} \langle L_V(\lambda)\rho, V \rangle_2 - \frac{1}{2\pi} \langle (\text{Id} + L_V(\lambda))^{-1}L_V(\lambda)\rho, L_V(\lambda)^*V \rangle_2. \end{aligned}$$

The first term is $O(\varepsilon^\infty)$. We note that since $\lambda \in [1, 2]$, $L_V(\lambda)^* = L_V(\lambda)$ and the third term can be bounded using the Cauchy–Schwarz inequality:

$$|\langle (\text{Id} + L_V(\lambda))^{-1}L_V(\lambda)\rho, L_V(\lambda)V \rangle_2| \leq |(\text{Id} + L_V(\lambda))^{-1}|_{\mathcal{B}} |L_V(\lambda)\rho|_2 |L_V(\lambda)V|_2. \quad (3.5)$$

We note that $L_V(\lambda)\rho = L_\rho(\lambda)V$ and we recall that $L_V(\lambda)$ is bounded from H^{-2} to L^2 . Hence Lemma 3.1 implies that $|L_V(\lambda)\rho|_2 \leq C|V|_{H^{-2}} = O(\varepsilon^2)$. Similarly,

$$|L_V(\lambda)V|_2 \leq C|L_\rho(\lambda)V| = O(\varepsilon^2)$$

This proves that the RHS of (3.5) is $O(\varepsilon^4)$. The steps below are devoted to estimating the leading term in the expansion of $\varphi_V(\lambda)$: $\langle L_V(\lambda)\rho, V \rangle_2$.

3. Using $L_V(\lambda) = K_V(\lambda) - \Pi_V \ln(\lambda)$, we have uniformly on $[1, 2]$

$$\begin{aligned} \langle L_V(\lambda)\rho, V \rangle_2 &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} R_0(\lambda, x, y) V(y) V(x) dx dy + \frac{\ln(\lambda)}{2\pi} \left(\int_{\mathbb{R}^2} V(x) dx \right)^2 \\ &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \phi_\lambda(x - y) V(y) V(x) dx dy + O(\varepsilon^\infty). \end{aligned} \quad (3.6)$$

Here the function ϕ_λ has Fourier transform equal to $\widehat{\phi_\lambda}(\xi) = (|\xi|^2 + \lambda^2)^{-1}$. The identity (3.6) and the Plancherel formula show that $\langle L_V(\lambda)\rho, V \rangle_2$ is equal modulo $O(\varepsilon^\infty)$ to

$$\langle \widehat{\phi_\lambda} \widehat{V}, \widehat{V} \rangle_2 = \int_{\mathbb{R}^2} \frac{\widehat{V}(\xi) \widehat{V}(-\xi)}{|\xi|^2 + \lambda^2} d\xi.$$

As the Fourier transform of V is given by $\sum_{k \neq 0} \widehat{W}_k(\xi - k/\varepsilon)$ we obtain

$$\langle L_V(\lambda)\rho, V \rangle_2 = \sum_{k, \ell} I[W_k, W_\ell] + O(\varepsilon^\infty), \quad I[W_k, W_\ell] \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{\widehat{W}_k(\xi - k/\varepsilon) \widehat{W}_\ell(-\xi + \ell/\varepsilon)}{|\xi|^2 + \lambda^2} d\xi.$$

We will study $I[W_k, W_\ell]$ depending whether $k + \ell \neq 0$ or $k + \ell = 0$.

4. Assume that $k + \ell \neq 0$. We have

$$\begin{aligned} |I[W_k, W_\ell]| &\leq C |W_k|_{C^3} |W_\ell|_{C^3} \int_{\mathbb{R}^2} \langle \xi \rangle^{-2} \langle \xi - k/\varepsilon \rangle^{-3} \langle -\xi + \ell/\varepsilon \rangle^{-3} d\xi \\ &\leq C \varepsilon^{5/2} \frac{|W_k|_{C^3} |W_\ell|_{C^3}}{|k - \ell|^{5/2}} \int_{\mathbb{R}^2} \langle \xi \rangle^{-2} \langle \xi - k/\varepsilon \rangle^{-1/2} \langle -\xi + \ell/\varepsilon \rangle^{-1/2} d\xi \end{aligned}$$

To get the second line, we controlled $\langle \xi - k/\varepsilon \rangle^{-5/2} \langle -\xi + \ell/\varepsilon \rangle^{-5/2}$ by $\langle (k - \ell)/\varepsilon \rangle^{-5/2}$ with Peetre's inequality (3.1). Hölder's inequality yields

$$|I[W_k, W_\ell]| \leq C \varepsilon^{5/2} \frac{|W_k|_{C^3} |W_\ell|_{C^3}}{|k - \ell|^{5/2}}.$$

5. We now focus on the case $k + \ell = 0$. A substitution yields

$$I[W_k, W_{-k}] = \varepsilon^2 \int_{\mathbb{R}^2} \frac{\widehat{W}_k(\xi) \widehat{W}_{-k}(-\xi)}{|\varepsilon \xi + k|^2 + (\varepsilon \lambda)^2} d\xi.$$

We split the domain of integration \mathbb{R}^2 into two parts: $\mathbb{B}(0, \varepsilon^{-3/4})$ and $\mathbb{R}^2 \setminus \mathbb{B}(0, \varepsilon^{-3/4})$. If $\xi \in \mathbb{R}^2 \setminus \mathbb{B}(0, \varepsilon^{-3/4})$ then $|\varepsilon \xi + k| + (\varepsilon \lambda)^2 \geq (\varepsilon \lambda)^2 \geq \varepsilon^2$ since $\lambda \in [1, 2]$. Therefore

$$\begin{aligned} \left| \int_{\mathbb{R}^2 \setminus \mathbb{B}(0, \varepsilon^{-1/2})} \frac{\widehat{W}_k(\xi) \widehat{W}_{-k}(-\xi)}{|\varepsilon \xi + k|^2 + (\varepsilon \lambda)^2} d\xi \right| &\leq \int_{|\xi| \geq \varepsilon^{-3/4}} \frac{1}{\varepsilon^2} |\widehat{W}_k(\xi) \widehat{W}_{-k}(-\xi)| d\xi \\ &\leq C \int_{|\xi| \geq \varepsilon^{-3/4}} \frac{1}{\varepsilon^2} \langle \xi \rangle^{-6} |W_k|_{C^3} |W_{-k}|_{C^3} d\xi \leq C \varepsilon |W_k|_{C^3} |W_{-k}|_{C^3}. \end{aligned}$$

We now assume that $\xi \in \mathbb{B}(0, \varepsilon^{-3/4})$, hence $|\varepsilon \xi + k| \geq |k| - \varepsilon^{1/4} \geq |k|/2$. This implies

$$\frac{1}{|\varepsilon \xi + k|^2 + (\varepsilon \lambda)^2} - \frac{1}{|k|^2} = \frac{-(\varepsilon \lambda)^2 - \varepsilon^2 |\xi|^2 - 2\langle \varepsilon \xi, k \rangle}{|k|^2 (|\varepsilon \xi + k|^2 + (\varepsilon \lambda)^2)} = O(\varepsilon^{1/4} |k|^{-3}).$$

This implies

$$\begin{aligned} \int_{\mathbb{B}(0, \varepsilon^{-1/2})} \frac{\widehat{W}_k(\xi) \widehat{W}_{-k}(-\xi)}{|\varepsilon \xi + k|^2 + (\varepsilon \lambda)^2} d\xi &= \int_{\mathbb{B}(0, \varepsilon^{-1/2})} \frac{\widehat{W}_k(\xi) \widehat{W}_{-k}(-\xi)}{|k|^2} d\xi + O(\varepsilon^{1/4}) \frac{|W_k|_{C^2} |W_{-k}|_{C^2}}{|k|^3} \\ &= \int_{\mathbb{R}^2} \frac{\widehat{W}_k(\xi) \widehat{W}_{-k}(-\xi)}{|k|^2} d\xi + O(\varepsilon^{1/4}) \frac{|W_k|_{C^2} |W_{-k}|_{C^2}}{|k|^3}. \end{aligned}$$

6. Combine the results of steps 1-5 to get

$$\begin{aligned}\varepsilon^{-2}\varphi_V(\lambda) &= \frac{1}{2\pi} \sum_{k \neq 0} I[W_k, W_{-k}] + o(1) = \frac{1}{2\pi} \sum_{k \neq 0} \frac{1}{|k|^2} \int_{\mathbb{R}^2} \widehat{W}_k(\xi) \widehat{W}_{-k}(-\xi) d\xi + o(1) \\ &= \frac{1}{2\pi} \sum_{k \neq 0} \frac{1}{|k|^2} \langle W_k, \overline{W_{-k}} \rangle_2 + o(1) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_0(x) dx + o(1), \quad \Lambda_0(x) = \sum_{k \neq 0} \frac{|W_k(x)|^2}{|k|^2}.\end{aligned}$$

In the above we used that as V is real-valued, $W_k = \overline{W_{-k}}$. This estimate completes the proof of the lemma. \square

We finally turn to the proof of Theorem 1.

Proof of Theorem 1. By the discussion following the proof of Lemma 3.2, for ε small enough the negative eigenvalues of $-\Delta + V$ are all in a fixed compact set and are the negative squares of the resonances of V in K_0 . We now work for $\lambda \in K_0$.

For ε small enough, $\text{Id} + L_V(\lambda)$ is invertible by Lemma 3.2. Lemma 2.2 shows that the resonances of V are the zeros of $1 + \ln(\lambda)\varphi_V(\lambda)$. Since $\varphi_V(\lambda) = \frac{\varepsilon^2}{2\pi} \int_{\mathbb{R}^2} \Lambda_0 + o(\varepsilon^2)$ uniformly on $K_0 = [0, M]$, if ε is small enough then

$$\lim_{\lambda \rightarrow 0} 1 + \ln(\lambda)\varphi_V(\lambda) = -\infty, \quad 1 + \ln(M)\varphi_V(M) \geq \frac{1}{2}.$$

We recall that φ_V is real-valued on K_0 . The intermediate value theorem shows that $1 + \ln(\lambda)\varphi_V(\lambda)$ has at least one zero λ on $[0, M]$. It must satisfy

$$\ln(\lambda) = -\frac{1}{\varphi_V(\lambda)} = -\frac{2\pi}{\varepsilon^2 \int_{\mathbb{R}^2} \Lambda_0(x) dx + o(\varepsilon^2)}, \quad (3.7)$$

or equivalently,

$$\lambda = \exp\left(-\frac{2\pi}{\varepsilon^2 \int_{\mathbb{R}^2} \Lambda_0(x) dx + o(\varepsilon^2)}\right).$$

It remains to show that λ is the unique resonance of V in K_0 for ε small enough. We argue as in [Si76, Theorem 2.3]. If $\mu \neq \lambda$ is another resonance in K_0 then $1 + \ln(\mu)\varphi_V(\mu) = 0$. Hence,

$$\left| \frac{1}{\ln(\mu)} - \frac{1}{\ln(\lambda)} \right| = |\varphi_V(\mu) - \varphi_V(\lambda)|. \quad (3.8)$$

In addition, for ε small enough, $|\ln(\lambda)|^{-1} \geq c\varepsilon^2$ and $|\ln(\mu)|^{-1} \geq c\varepsilon^2$. Hence, a lower bound for the right hand side of (3.8) is

$$\left| \frac{1}{\ln(\mu)} - \frac{1}{\ln(\lambda)} \right| \geq \frac{1}{|\ln(\lambda)||\ln(\mu)|} \left| \int_{\lambda}^{\mu} \frac{dt}{t} \right| \geq \frac{|\lambda - \mu|}{|\ln(\lambda)||\ln(\mu)||\lambda + \mu|} \geq \frac{c^2 \varepsilon^4 |\lambda - \mu|}{|\lambda + \mu|}.$$

An upper bound for the left hand side of (3.8) is provided by Lemma 3.3: $|\varphi_V(\lambda) - \varphi_V(\mu)| \leq C|\lambda \ln(\lambda) - \mu \ln(\mu)|$. Therefore, (3.8) implies

$$\frac{c^2 \varepsilon^4}{|\lambda + \mu|} \leq C \frac{|\lambda \ln(\lambda) - \mu \ln(\mu)|}{|\lambda - \mu|}.$$

Since the derivative of $t \ln(t)$ is $1 + \ln(t)$, and since λ, μ remain in a bounded set with $\ln(\lambda) = O(\varepsilon^{-2})$ (and a similar bound for μ), we deduce that $\varepsilon^4 \leq C\varepsilon^{-2}|\lambda + \mu|$. In addition, $|\ln(\lambda)|^{-1} \geq c\varepsilon^2$ implies $|\lambda| \leq e^{-c^{-1}\varepsilon^{-2}}$ (and a similar bound for μ). Hence, the existence of μ implies $c^2\varepsilon^4 \leq 2C\varepsilon^{-2}e^{-c^{-1}\varepsilon^{-2}} = O(\varepsilon^\infty)$. This is a contradiction. \square

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